Please check the examination details below before entering your candidate information

Candidate surname	Other names

Centre Number | Candidate Number

Pearson Edexcel Level 3 GCE

Time 2 hours

Paper reference **9MA0/02**

Mathematics

Advanced

PAPER 2: Pure Mathematics 2

You must have:
Mathematical Formulae and Statistical Tables (Green), calculator

Total Marks

Candidates may use any calculator allowed by Pearson regulations. Calculators must not have the facility for symbolic algebra manipulation, differentiation and integration, or have retrievable mathematical formulae stored in them.

Instructions

- Use **black** ink or ball-point pen.
- If pencil is used for diagrams/sketches/graphs it must be dark (HB or B).
- **Fill in the boxes** at the top of this page with your name, centre number and candidate number.
- Answer **all** questions and ensure that your answers to parts of questions are clearly labelled.
- Answer the questions in the spaces provided – *there may be more space than you need.*
- You should show sufficient working to make your methods clear. Answers without working may not gain full credit.
- Inexact answers should be given to three significant figures unless otherwise stated.

Information

- A booklet 'Mathematical Formulae and Statistical Tables' is provided.
- There are 16 questions in this question paper. The total mark for this paper is 100.
- The marks for **each** question are shown in brackets – *use this as a guide as to how much time to spend on each question.*

Advice

- Read each question carefully before you start to answer it.
- Try to answer every question.
- Check your answers if you have time at the end.

Turn over

1. **In this question you must show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

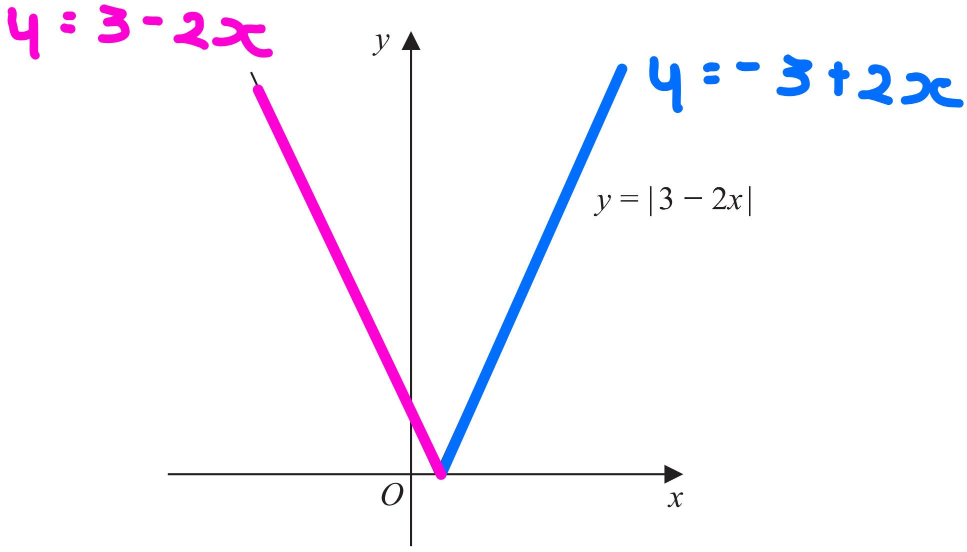

Figure 1

Figure 1 shows a sketch of the graph with equation $y = |3 - 2x|$

Solve

$$|3 - 2x| = 7 + x$$

(4)

Case 1:

$3 - 2x = 7 + x$

$-3x = 4$

$x = -\frac{4}{3}$

Case 2:

$-3 + 2x = 7 + x$

$x = 10$

$x = -\frac{4}{3}, x = 10$

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2. (a) Sketch the curve with equation

$$y = 4^x$$

stating any points of intersection with the coordinate axes.

(2)

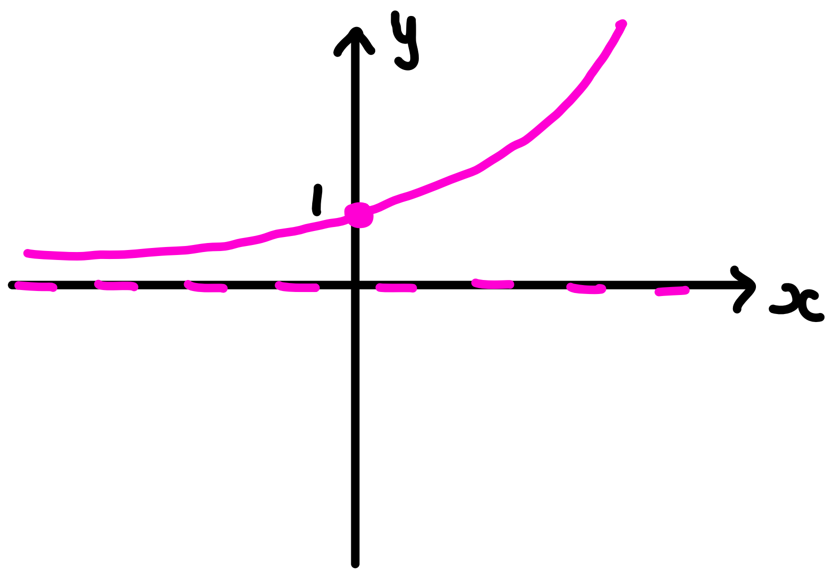
(b) Solve

$$4^x = 100$$

giving your answer to 2 decimal places.

(2)

a)



b) $4^x = 100$

$\log 4^x = \log 100$

$x \log 4 = \log 100$

$x = \dfrac{\log 100}{\log 4}$

$x = 3.32$

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3. A sequence of terms $a_1, a_2, a_3, \ldots$ is defined by

$$a_1 = 3$$
$$a_{n+1} = 8 - a_n$$

(a) (i) Show that this sequence is periodic.

(ii) State the order of this periodic sequence.

(2)

(b) Find the value of

$$\sum_{n=1}^{85} a_n$$

(2)

$a_1 = 3$
$a_{n+1} = 8 - a_n$

a)
i) $a_1 = 3$

$a_2 = 8 - a_1 = 8 - 3 = 5$

$a_3 = 8 - a_2 = 8 - 5 = 3$

$a_4 = 8 - a_3 = 8 - 3 = 5$

sequence looks like 3, 5, 3, 5 ...
hence repeats every 2 'terms' and
is periodic

ii) order 2

b) $\sum_{n=1}^{85} a_n$

This wants the sum of the first 85
terms

$\underbrace{3+5}$ + 3 + 5 + 3 + 5 + ...

Sum = 8

Question 3 continued

$\frac{85}{2} = 42$ remainder 1

So we have 42 "lots" of the sum 8 plus 1 term

$42(8) = 336$

$336 + 3 = 339$

(Total for Question 3 is 4 marks)

4. Given that

$$y = 2x^2$$

use differentiation from first principles to show that

$$\frac{dy}{dx} = 4x$$

(3)

$f(x) = 2x^2$

$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$= \lim_{h \to 0} \frac{2(x+h)^2 - 2x^2}{h}$

$= \lim_{h \to 0} \frac{2(x^2+2xh+h^2) - 2x^2}{h}$

$= \lim_{h \to 0} \frac{2x^2+4xh+2h^2-2x^2}{h}$

$= \lim_{h \to 0} \frac{4xh+h^2}{h}$

$= \lim_{h \to 0} \frac{h(4x+h)}{h}$

$= \lim_{h \to 0} (4x-h)$

$= 4x$

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5. The table below shows corresponding values of x and y for $y = \log_3 2x$

The values of y are given to 2 decimal places as appropriate.

x	3	4.5	6	7.5	9
y	1.63	2	2.26	2.46	2.63
	y_0	y_1	y_2	y_3	y_4

(a) Using the trapezium rule with all the values of y in the table, find an estimate for

$$\int_3^9 \log_3 2x \, dx$$

(3)

Using your answer to part (a) and making your method clear, estimate

(b) (i) $\int_3^9 \log_3 (2x)^{10} \, dx$

(ii) $\int_3^9 \log_3 18x \, dx$

(3)

a) $h = \frac{9-3}{4} = \frac{6}{4} = \frac{3}{2}$

Formula: $\frac{h}{2}\left[y_0 + 2(y_1 + y_2 + y_3) + y_4\right]$

$= \frac{\frac{3}{2}}{2}\left[1.63 + 2(2 + 2.26 + 2.46) + 2.63\right]$

$= \frac{3}{4}\left[1.63 + 2(2 + 2.26 + 2.46) + 2.63\right]$

$= 13.275$

Question 5 continued

b)

i) $\int_3^9 \log_3(2x)^{10}\,dx$

$= \int_3^9 10\log_3 2x\,dx$

$= 10\int_3^9 \underbrace{\log_3 2x\,dx}_{=13.275 \text{ from a)}}$

$= 10(13.275)$

$= 132.75$

ii) $\int_3^9 \log_3 18x\,dx$

$= \int_3^9 \log_3(9\times 2x)\,dx$

$= \int_3^9 (\log_3 9 + \log_3 2x)\,dx$

$= \int_3^9 (2 + \log_3 2x)\,dx$

$= \int_3^9 2\,dx + \underbrace{\int_3^9 \log_3 2x\,dx}_{=13.275 \text{ from a)}}$

$= \left[2x\right]_3^9 + 13.275$

$= 18 - 6 + 13.275 = 25.275$

(Total for Question 5 is 6 marks)

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6.

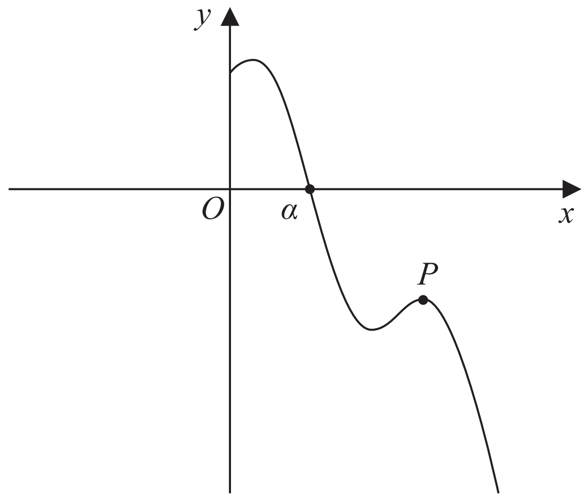


Figure 2

Figure 2 shows a sketch of part of the curve with equation $y = \mathrm{f}(x)$ where

$$\mathrm{f}(x) = 8\sin\left(\frac{1}{2}x\right) - 3x + 9 \qquad x > 0$$

and x is measured in radians.

The point P, shown in Figure 2, is a local maximum point on the curve.

Using calculus and the sketch in Figure 2,

(a) find the x coordinate of P, giving your answer to 3 significant figures.

(4)

The curve crosses the x-axis at $x = \alpha$, as shown in Figure 2.

Given that, to 3 decimal places, $\mathrm{f}(4) = 4.274$ and $\mathrm{f}(5) = -1.212$

(b) explain why α must lie in the interval $[4, 5]$

(1)

(c) Taking $x_0 = 5$ as a first approximation to α, apply the Newton-Raphson method once to $\mathrm{f}(x)$ to obtain a second approximation to α.

Show your method and give your answer to 3 significant figures.

(2)

a) max when $f'(x) = 0$

$f'(x) = 8\left(\frac{1}{2}\right)\cos\left(\frac{1}{2}x\right) - 3$

$= 4\cos\left(\frac{1}{2}x\right) - 3$

$\Rightarrow 4\cos\left(\frac{1}{2}x\right) - 3 = 0$

$\Rightarrow \cos\left(\frac{1}{2}x\right) = \frac{3}{4}$

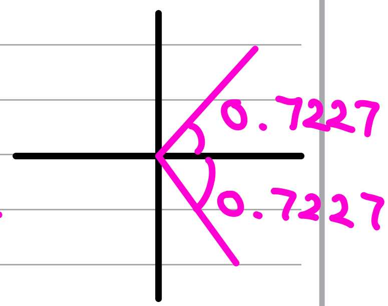
$\frac{1}{2}x = 0.7227, 5.5605, 7.006, \ldots$

$x = 1.445, 11.121, 14.012$

We want the 3rd solution
(the 1st two give the 1st max & min)

$x = 14.0$

$\cos^{-1}\left(\frac{3}{4}\right) = 0.7227$



$= 0.7229$
$= 2\pi - 0.7229$
$= 2\pi + 0.7229$
etc

b) crosses x axis when $f(x) = 0$

α must lie in $[4, 5]$ since $f(x)$ has a sign change in this interval and $f(x)$ is continuous

c) $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

$x_1 = x_0 - \frac{f(x_1)}{f'(x_1)}$

$= 5 - \frac{8\sin\left(\frac{1}{2} \times 5\right) - 3(5) + 9}{4\cos\left(\frac{1}{2} \times 5\right) - 3}$

$= 5 - \frac{8\sin(2.5) - 6}{4\cos(2.5) - 3}$

$= 5 - \frac{-1.21222}{-6.20457}$

$= 4.80$

7. (a) Find the first four terms, in ascending powers of x, of the binomial expansion of

$$\sqrt{4-9x}$$

writing each term in simplest form.

(4)

A student uses this expansion with $x = \frac{1}{9}$ to find an approximation for $\sqrt{3}$

Using the answer to part (a) and without doing any calculations,

(b) state whether this approximation will be an overestimate or an underestimate of $\sqrt{3}$ giving a brief reason for your answer.

(1)

7) $(4-9x)^{\frac{1}{2}}$

First we need to get into the form $(1 \pm \ldots)^{\frac{1}{2}}$

$(4-9x)^{\frac{1}{2}} = \left(4\left[1-\frac{9x}{4}\right]\right)^{\frac{1}{2}}$

$= 4^{\frac{1}{2}}\left(1-\frac{9x}{4}\right)^{\frac{1}{2}}$

$= 2\underbrace{\left(1-\frac{9x}{4}\right)^{\frac{1}{2}}}$ 1

lets find the expansion for this

Use formula $1+nx+\frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3+\ldots$

Here $n=\frac{1}{2}$ and $x \rightarrow -\frac{9x}{4}$

Question 7 continued

$$1+\left(\frac{1}{2}\right)\left(\frac{-9x}{4}\right)+\frac{\left(\frac{1}{2}\right)\left(\frac{-1}{2}\right)}{2!}\left(\frac{-9x}{4}\right)^2+\frac{\left(\frac{1}{2}\right)\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)}{3!}\left(\frac{-9x}{4}\right)^3+\ldots$$

$$=1-\frac{9}{8}x+\frac{\left(\frac{1}{2}\right)\left(\frac{-1}{2}\right)\left(\frac{81}{16}\right)}{2}x^2+\frac{\left(\frac{1}{2}\right)\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)\left(\frac{-729}{64}\right)}{6}x^3+\ldots$$

$$=1-\frac{9}{8}x-\frac{81}{128}x^2-\frac{729}{1024}x^3+\ldots$$

Don't forget about the 2 in 1

$$2\left(1-\frac{9}{8}x-\frac{81}{128}x^2-\frac{729}{1024}x^3+\ldots\right)$$

$$=2-\frac{9}{4}x-\frac{81}{64}x^2-\frac{729}{512}x^3+\ldots$$

b) overestimate since every additional term (+...) is negative. So, the approximation will decrease with each additional term

(Total for Question 7 is 5 marks)

8. **In this question you must show all stages of your working.**

Solutions relying on calculator technology are not acceptable.

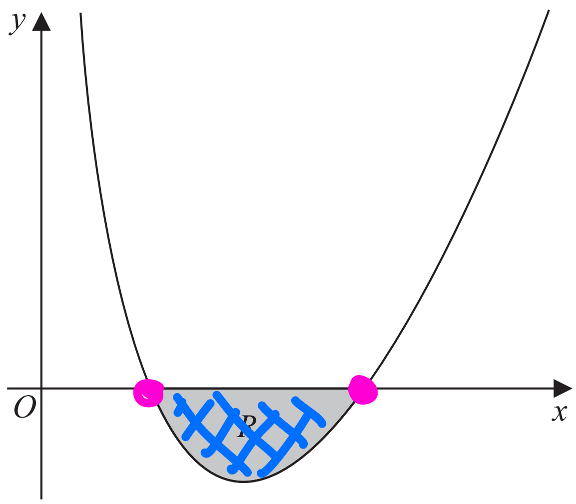


Figure 3

Figure 3 shows a sketch of part of a curve with equation

$$y = \frac{(x-2)(x-4)}{4\sqrt{x}} \qquad x > 0$$

The region R, shown shaded in Figure 3, is bounded by the curve and the x-axis.

Find the exact area of R, writing your answer in the form $a\sqrt{2} + b$, where a and b are constants to be found.

(6)

To find • points set $y = 0$:

$$\frac{(x-2)(x-4)}{4\sqrt{x}} = 0$$

$$\Rightarrow (x-2)(x-4) = 0$$

$$\Rightarrow x = 2, \; x = 4$$

$$\text{Area} = \int_2^4 \frac{(x-2)(x-4)}{4\sqrt{x}}\,dx$$

$$= \int_2^4 \frac{x^2 - 6x + 8}{4x^{\frac{1}{2}}}\,dx$$

Alternative method: take $\frac{1}{4}$ out

$$\frac{1}{4}\int_2^4 \frac{x^2 - 6x + 8}{x^{\frac{1}{2}}}\,dx$$

$$= \int_2^4 \left(\frac{x^2}{4x^{\frac{1}{2}}} - \frac{6x}{4x^{\frac{1}{2}}} + \frac{8}{4x^{\frac{1}{2}}}\right) dx$$

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Question 8 continued

$$= \int_2^4 \left(\frac{1}{4}x^{2-\frac{1}{2}} - \frac{3}{2}x^{1-\frac{1}{2}} + 2x^{-\frac{1}{2}}\right) dx$$

$$= \int_2^4 \left(\frac{1}{4}x^{\frac{3}{2}} - \frac{3}{2}x^{\frac{1}{2}} + 2x^{-\frac{1}{2}}\right) dx$$

$$= \left[\frac{1}{4}\frac{x^{\frac{5}{2}}}{\frac{5}{2}} - \frac{3}{2}\frac{x^{\frac{3}{2}}}{\frac{3}{2}} + \frac{2x^{\frac{1}{2}}}{\frac{1}{2}}\right]_2^4$$

$$= \left[\frac{1}{10}x^{\frac{5}{2}} - x^{\frac{3}{2}} + 4x^{\frac{1}{2}}\right]_2^4$$

$$= \left(\frac{1}{10}(4)^{\frac{5}{2}} - (4)^{\frac{3}{2}} + 4(4)^{\frac{1}{2}}\right) - \left(\frac{1}{10}(2)^{\frac{5}{2}} - (2)^{\frac{3}{2}} + 4(2)^{\frac{1}{2}}\right)$$

$$= \left(\frac{1}{10}(32) - (8) + 4(2)\right) - \left(\frac{1}{10}(\sqrt{2})^5 - (\sqrt{2})^3 + 4\sqrt{2}\right)$$

$$= \left(\frac{32}{10} - 8 + 8\right) - \left(\frac{1}{10}(4\sqrt{2}) - 2\sqrt{2} + 4\sqrt{2}\right)$$

$$= \left(\frac{32}{10}\right) - \left(\frac{2}{5}\sqrt{2} + 2\sqrt{2}\right)$$

$$= \frac{32}{10} - \frac{12}{5}\sqrt{2} = \frac{16}{5} - \frac{12}{5}\sqrt{2}$$

Area is positive so take positive version i.e multiply by -1

$\frac{12}{5}\sqrt{2} - \frac{16}{5} \Rightarrow a = \frac{12}{5}, b = -\frac{16}{5}$

9.

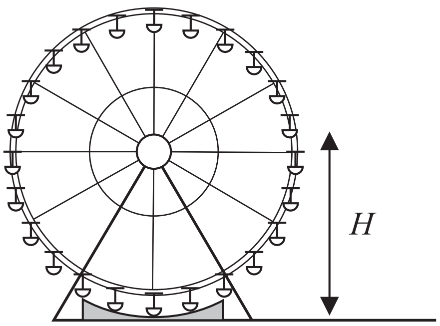

Figure 4

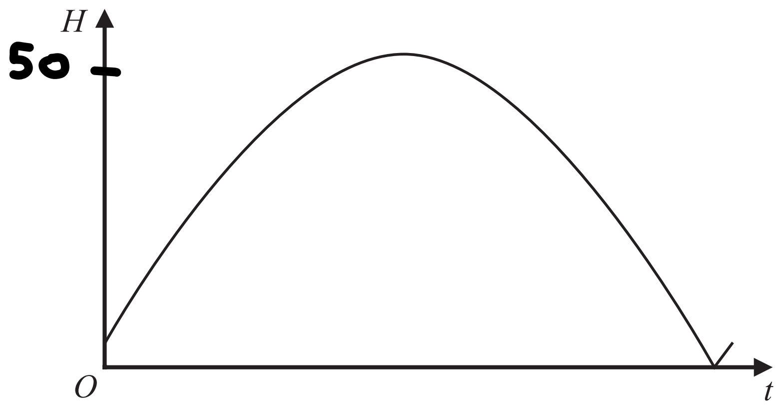

Figure 5

Figure 4 shows a sketch of a Ferris wheel.

The height above the ground, H m, of a passenger on the Ferris wheel, t seconds after the wheel starts turning, is modelled by the equation

$$H = |A\sin(bt + \alpha)°|$$

where A, b and α are constants.

Figure 5 shows a sketch of the graph of H against t, for one revolution of the wheel.

Given that

- the maximum height of the passenger above the ground is 50 m
- the passenger is 1 m above the ground when the wheel starts turning
- the wheel takes 720 seconds to complete one revolution

(a) find a complete equation for the model, giving the exact value of A, the exact value of b and the value of α to 3 significant figures.

(4)

(b) Explain why an equation of the form

$$H = |A\sin(bt + \alpha)°| + d$$

where d is a positive constant, would be a more appropriate model.

(1)

a) $H = |A\sin(bt + \alpha)°|$

Amplitude $= A = \frac{\max - \min}{2} = \frac{50 - -50}{2} = 50$

period $= \frac{180}{b} \Rightarrow 720 = \frac{180}{b} \Rightarrow b = \frac{1}{4}$

Question 9 continued

Fill A and b into the equation:

$H = \left| 50\sin\left(\frac{1}{4}t + \alpha\right)^{\circ} \right|$

When $t = 0, h = 1$:

$1 = |50\sin\alpha|$

$1 = 50\sin\alpha$

$\sin\alpha = \frac{1}{50}$

$\alpha = \sin^{-1}\left(\frac{1}{50}\right) = 1.146$

$H = \left| 50\sin\left(\frac{1}{4}t + 1.15\right)^{\circ} \right|$

b) The existing model has the passenger starting 1 m above the ground, but there is a point where the are 0 m above the ground. In reality the passenger would never be on the ground. This is still an awful model though! Should be $H = A\sin(bt + \alpha) + d$ without the modulus. Stupid question! Putting the modulus would only be accurate for the height of a person if the Ferris wheel broke free from its mounting and was rolling along the ground & the cars were dragging along the ground

(Total for Question 9 is 5 marks)

10. The function f is defined by

$$f(x) = \frac{8x + 5}{2x + 3} \qquad x > -\frac{3}{2}$$

(a) Find $f^{-1}\left(\frac{3}{2}\right)$

(2)

(b) Show that

$$f(x) = A + \frac{B}{2x + 3}$$

where A and B are constants to be found.

(2)

The function g is defined by

$$g(x) = 16 - x^2 \qquad 0 \leqslant x \leqslant 4$$

(c) State the range of g^{-1}

(1)

(d) Find the range of fg^{-1}

(3)

$f(x) = \frac{8x+5}{2x+3}$

a) $y = \frac{8x+5}{2x+3}$

$x = \frac{8y+5}{2y+3}$

$x(2y+3) = 8y+5$

$2xy + 3x = 8y+5$

$2xy - 8y = 5 - 3x$

$y(2x-8) = 5-3x$

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$y = \frac{5-3x}{2x-8} = f^{-1}(x)$

$f^{-1}\left(\frac{3}{2}\right) = \frac{5-3\left(\frac{3}{2}\right)}{2\left(\frac{3}{2}\right)-8} = \frac{5-\frac{9}{2}}{3-8} = \frac{0.5}{-5} = -0.1$

b) Need to divide to get into this form

$$2x+3 \overline{)8x+5} \quad 4$$
$$8x+12$$
$$-7$$

$f(x) = 4 - \frac{7}{2x+3}$

$A = 4,\ B = -7$

c) $g(x) = 16 - x^2,\ 0 \leq x \leq 4$

The range of $g^{-1}(x)$ is the domain of g

Range: $0 \leq y \leq 4$

d) $f(x) = \frac{8x+5}{2x+3},\ x > -\frac{3}{2}$

$g(x) = 16 - x^2,\ 0 \leq x \leq 4$

Need to find $g^{-1}(x)$

Way 2: Without even finding $fg^{-1}(x)$

The range of g^{-1} will provide the domain of the

Question 10 continued

$y = 16 - x^2$

$x = 16 - y^2$

$y^2 = 16 - x$

$y = \sqrt{16 - x}$

$g^{-1}(x) = \sqrt{16 - x}$

$fg^{-1}(x) = \dfrac{8\sqrt{16-x}+5}{2\sqrt{16-x}+3}$

f has domain $x > -\frac{3}{2}$

g^{-1} has domain $0 \leq x \leq 16$ since $g(x)$ has domain $0 \leq x \leq 4$

so overlap is $0 \leq x \leq 16$

When $x = 0$: $fg^{-1}(x) = \dfrac{8(4)+5}{2(4)+3} = \dfrac{37}{11}$

When $x = 16$: $fg^{-1}(x) = \dfrac{8(0)+5}{2(0)+3} = \dfrac{5}{3}$

Range fg^{-1} is $\frac{5}{3} \leq x \leq \frac{37}{11}$

Need to look at range of outside function too since composite

$f(x) = \dfrac{8x+5}{2x+3}$, $x > -\frac{3}{2}$

Also told $0 \leq x \leq 4$

$x = 0$: $f(x) = \frac{5}{3}$

$x = 4$: $f(x) = \frac{37}{11}$

Range of f is $\frac{5}{3} \leq x \leq \frac{37}{11}$

take overlap of gives $\frac{5}{3} \leq x \leq \frac{37}{11}$

composite part of the function f

$f(0) = \dfrac{0+5}{0+3} = \dfrac{5}{3}$

$f(4) = \dfrac{32+5}{8+3} = \dfrac{37}{11}$

$\frac{5}{3} \leq y \leq \frac{37}{11}$

11. Prove, using algebra, that

$$n(n^2 + 5)$$

is even for all $n \in \mathbb{N}$.

(4)

Numbers are either even or odd and hence look like $2k$ or $2k+1$ where $k \in \mathbb{Z}$

Case ①
$n = 2k$: $2k[(2k)^2 + 5]$

$= 2k(4k^2 + 5)$

$= 8k^3 + 10k$

$= 2(4k^3 + 5k)$

multiple of 2 hence even

Case ②:
$n = 2k+1$
$(2k+1)[(2k+1)^2 + 5]$

$= (2k+1)(4k^2 + 4k + 6)$

$= 8k^3 + 8k^2 + 12k + 4k^2 + 4k + 6$

$= 8k^3 + 12k^2 + 16k + 6$

$= 2(4k^3 + 6k^2 + 8k + 3)$

multiple of 2 hence even

$\therefore$ even for all $n \in \mathbb{N}$

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12. The function f is defined by

$$f(x) = \frac{e^{3x}}{4x^2 + k}$$

where k is a positive constant.

(a) Show that

$$f'(x) = (12x^2 - 8x + 3k)\,g(x)$$

where $g(x)$ is a function to be found.

(3)

Given that the curve with equation $y = f(x)$ has at least one stationary point,

(b) find the range of possible values of k.

(3)

a) $f(x) = \dfrac{e^{3x}}{4x^2+k}$

Use quotient rule

$f'(x) = \dfrac{(4x^2+k)\,3e^{3x} - e^{3x}(8x)}{(4x^2+k)^2}$

$= \dfrac{e^{3x}\left[3(4x^2+k) - 8x\right]}{(4x^2+k)^2}$

$= \dfrac{e^{3x}(12x^2 + 3k - 8x)}{(4x^2+k)^2}$

$= (12x^2 - 8x + 3k)\dfrac{e^{3x}}{(4x^2+k)^2}$

$\Rightarrow g(x) = \dfrac{e^{3x}}{(4x^2+k)^2}$

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Question 12 continued

b) solve $g'(x) = 0$

$$\frac{(12x^2 - 8x + 3k)e^{3x}}{(4x^2+k)^2} = 0$$

$$\underbrace{(12x^2 - 8x + 3k)}\underbrace{e^{3x}} = 0$$

$12x^2 - 8x + 3k = 0$ $\quad e^{3x} \neq 0$

$\downarrow$

told at least 1 solution (since at least 1 stationary point)

This means $b^2 - 4ac \geq 0$

$a = 12$
$b = -8$
$c = 3k$

$(-8)^2 - 4(12)(3k) \geq 0$

$64 - 144k \geq 0$

$144k \leq 64$

$k \leq \frac{4}{9}$

Also told k is positive

$0 \leq k \leq \frac{4}{9}$

(Total for Question 12 is 6 marks)

13. Relative to a fixed origin O

- the point A has position vector $4\mathbf{i} - 3\mathbf{j} + 5\mathbf{k}$
- the point B has position vector $4\mathbf{j} + 6\mathbf{k}$
- the point C has position vector $-16\mathbf{i} + p\mathbf{j} + 10\mathbf{k}$

where p is a constant.

Given that A, B and C lie on a straight line,

(a) find the value of p.

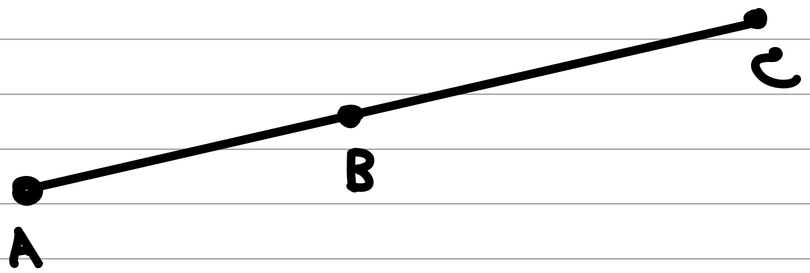
(3)

The line segment OB is extended to a point D so that $\overrightarrow{CD}$ is parallel to $\overrightarrow{OA}$

(b) Find $\left|\overrightarrow{OD}\right|$, writing your answer as a fully simplified surd.

(3)

a)



$$\overrightarrow{AB} = \begin{pmatrix} 0 \\ 4 \\ 6 \end{pmatrix} - \begin{pmatrix} 4 \\ -3 \\ 5 \end{pmatrix} = \begin{pmatrix} -4 \\ 7 \\ 1 \end{pmatrix}$$

$$\overrightarrow{BC} = \begin{pmatrix} -16 \\ p \\ 10 \end{pmatrix} - \begin{pmatrix} 0 \\ 4 \\ 6 \end{pmatrix} = \begin{pmatrix} -16 \\ p-4 \\ 4 \end{pmatrix}$$

$\overrightarrow{AB}$ and $\overrightarrow{BC}$ must be multiples since straight line

$$\begin{pmatrix} -4 \\ 7 \\ 1 \end{pmatrix} = k \begin{pmatrix} -16 \\ p-4 \\ 4 \end{pmatrix}$$

$$\begin{pmatrix} -4 \\ 7 \\ 1 \end{pmatrix} = \begin{pmatrix} -16k \\ pk-4k \\ 4k \end{pmatrix}$$

Question 13 continued

$-4 = -16k \Rightarrow k = \frac{1}{4}$

$7 = pk - 4k$

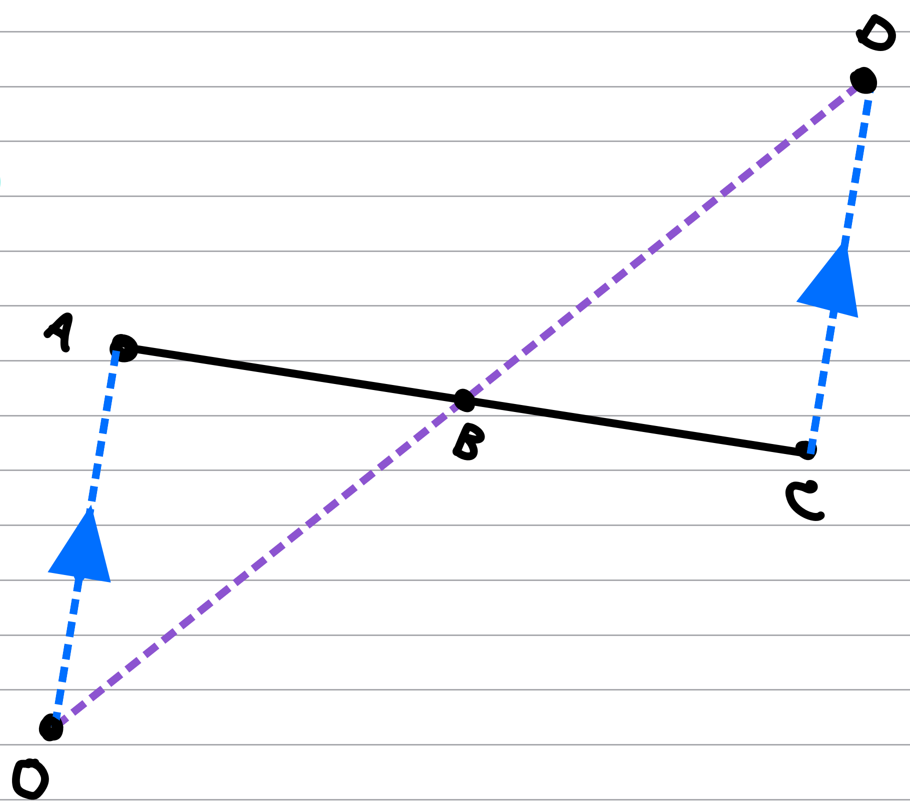
$7 = p\left(\frac{1}{4}\right) - 4\left(\frac{1}{4}\right)$

$7 = \frac{p}{4} - 1$

$\frac{p}{4} = 8$

$p = 32$

b)



$\overrightarrow{OA} = \begin{pmatrix} 4 \\ -3 \\ 5 \end{pmatrix}$, $\overrightarrow{CD} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} -16 \\ 32 \\ 10 \end{pmatrix} = \begin{pmatrix} x+16 \\ y-32 \\ z-10 \end{pmatrix}$

• $\overrightarrow{OA} = \overrightarrow{CD}$

$\begin{pmatrix} x+16 \\ y-32 \\ z-10 \end{pmatrix} = k \begin{pmatrix} 4 \\ -3 \\ 5 \end{pmatrix}$

$$\begin{pmatrix} x+16 \\ y-32 \\ z-10 \end{pmatrix} = \begin{pmatrix} 4k \\ -3k \\ 5k \end{pmatrix}$$

$x+16 = 4k \Rightarrow x = 4k-16$
$y-32 = -3k \Rightarrow y = -3k+32$ ①
$z-10 = 5k \Rightarrow z = 5k+10$

• We also know $\overrightarrow{OD} = \lambda \overrightarrow{OB}$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \lambda \begin{pmatrix} 0 \\ 4 \\ 6 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 4\lambda \\ 6\lambda \end{pmatrix}$$

$x = 0$
$y = 4\lambda$ ②
$z = 6\lambda$

equate the corresponding x, y and z compoments of ① and ②

$4k - 16 = 0$
$-3k + 32 = 4\lambda$] solving simult. gives $k = 4$, $\lambda = 5$
$5k + 10 = 6\lambda$

$$\overrightarrow{OD} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 4\lambda \\ 6\lambda \end{pmatrix} = \begin{pmatrix} 0 \\ 20 \\ 30 \end{pmatrix}$$

$$|\overrightarrow{OD}| = \sqrt{0^2 + 20^2 + 30^2} = \sqrt{1300} = 10\sqrt{13}$$

DO NOT WRITE IN THIS AREA

14. (a) Express $\dfrac{3}{(2x-1)(x+1)}$ in partial fractions.

(3)

When chemical A and chemical B are mixed, oxygen is produced.

A scientist mixed these two chemicals and measured the total volume of oxygen produced over a period of time.

The total volume of oxygen produced, $V\,\text{m}^3$, t hours after the chemicals were mixed, is modelled by the differential equation

$$\frac{dV}{dt} = \frac{3V}{(2t-1)(t+1)} \qquad V \geqslant 0 \qquad t \geqslant k$$

where k is a constant.

Given that exactly 2 hours after the chemicals were mixed, a total volume of $3\,\text{m}^3$ of oxygen had been produced,

(b) solve the differential equation to show that

$$V = \frac{3(2t-1)}{(t+1)}$$

(5)

The scientist noticed that

- there was a **time delay** between the chemicals being mixed and oxygen being produced
- there was a **limit** to the total volume of oxygen produced

Deduce from the model

(c) (i) the **time delay** giving your answer in minutes,

(ii) the **limit** giving your answer in m^3

(2)

a) $\dfrac{3}{(2x-1)(x+1)} = \dfrac{A}{2x-1} + \dfrac{B}{x+1} = \dfrac{A(x+1)+B(2x-1)}{(2x-1)(x+1)}$

Using cover up method:

let $x=-1$: $B(-2-1)=3$

$-3B=3$

$B=-1$

let $x=\frac{1}{2}$: $A(\frac{1}{2}+1)=3$

$\frac{3}{2}A=3$

$A=2$

Question 14 continued

$$\frac{2}{2x-1} + \frac{-1}{x+1} = \frac{2}{2x-1} - \frac{1}{x+1}$$

b) $\frac{dv}{dt} = \frac{3v}{(2t-1)(t+1)}$

$$3v\,dt = (2t-1)(t+1)\,dv$$

$$\int \frac{3}{(2t-1)(t+1)}\,dt = \int \frac{1}{v}\,dv$$

$$\int \left(\frac{2}{2t-1} - \frac{1}{t+1}\right) dt = \int \frac{1}{v}\,dv$$

$$\ln|2t-1| - \ln|t+1| = \ln v + c \quad ①$$

given when $t=2$, $v=3$:

$$\ln|4-1| - \ln|2+1| = \ln 3 + c$$

$$\ln 3 - \ln 3 = \ln 3 + c$$

$$c = -\ln 3$$

① becomes

$$\ln|2t-1| - \ln|t+1| = \ln v - \ln 3$$

$$\ln\left|\frac{2t-1}{t+1}\right| = \ln\frac{v}{3}$$

$$\Rightarrow \frac{2t-1}{t+1} = \frac{v}{3}$$

Question 14 continued

$\Rightarrow V = \dfrac{3(2t-1)}{t+1}$

c)

i) let $t=0$: $V = \dfrac{3(0-1)}{0+1} = -3$

We need $V>0$ since V can't be neg

So need $\dfrac{3(2t-1)}{t+1} \geq 0$

$3(2t-1) \geq 0$

$6t \geq 3$

$t \geq \frac{1}{2}$ so time delay is 30 mins

ii) $\lim\limits_{t\to\infty} \dfrac{3(2t-1)}{t+1}$

$= \lim\limits_{t\to\infty} \dfrac{\frac{6t}{t} - \frac{1}{t}}{\frac{t}{t} + \frac{1}{t}}$

$= \lim\limits_{t\to\infty} \dfrac{6 - \frac{1}{t}}{1+\frac{1}{t}}$

$= \dfrac{6-0}{1+0}$

$= 6\ m^3$

15. **In this question you must show all stages of your working.**

Solutions relying on calculator technology are not acceptable.

Given that the first three terms of a geometric series are

$$12\cos\theta \qquad 5 + 2\sin\theta \qquad \text{and} \qquad 6\tan\theta$$

(a) show that

$$4\sin^2\theta - 52\sin\theta + 25 = 0$$

(3)

Given that θ is an obtuse angle measured in radians,

(b) solve the equation in part (a) to find the exact value of θ

(2)

(c) show that the sum to infinity of the series can be expressed in the form

$$k(1 - \sqrt{3})$$

where k is a constant to be found.

(5)

$12\cos\theta,\ 5+2\sin\theta,\ 6\tan\theta$

a) $\dfrac{5+2\sin\theta}{12\cos\theta} = \dfrac{6\tan\theta}{5+2\sin\theta}$

$(5+2\sin\theta)(5+2\sin\theta) = 12\cos\theta(6\tan\theta)$

$25 + 20\sin\theta + 4\sin^2\theta = 72\cos\theta\dfrac{\sin\theta}{\cos\theta}$

$4\sin^2\theta + 20\sin\theta + 25 = 72\sin\theta$

$4\sin^2\theta - 52\sin\theta + 25 = 0$

b) let $y = \sin\theta$

$4y^2 - 52y + 25 = 0$

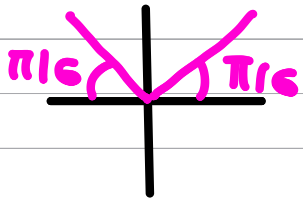
$(2y - 1)(2y - 25) = 0$

$y = \frac{1}{2},\ y = 12.5$

Question 15 continued

$\sin\theta = \frac{1}{2}$, $\sin\theta = 12.5$
No solution

$\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$



$\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \ldots$

given θ is obtuse

$\theta = \frac{5\pi}{6}$

c) sequence becomes:

$12\cos 150°,\ 5 + 2\sin 150°,\ 6\tan 150°$

$-6\sqrt{3},\ 6,\ -2\sqrt{3}$

$a = -6\sqrt{3}$

$r = \frac{6}{-6\sqrt{3}} = -\frac{1}{\sqrt{3}} = -\frac{\sqrt{3}}{3}$

$S_\infty = \frac{a}{1-r} = \frac{-6\sqrt{3}}{1+\frac{\sqrt{3}}{3}} = \frac{-6\sqrt{3}}{\frac{3+\sqrt{3}}{3}} = -6\sqrt{3} \times \frac{3}{3+\sqrt{3}}$

$= \frac{-18\sqrt{3}}{3+\sqrt{3}}$

Note: Instead can multiply all terms by 3

$\frac{-18\sqrt{3}}{3+\sqrt{3}}$

now rationalise

$\frac{-5+\sqrt{3}+5+}{9-3} = -9\sqrt{3}+9$

Question 15 continued

$$\frac{-18\sqrt{3}}{3+\sqrt{3}} \times \frac{3-\sqrt{3}}{3-\sqrt{3}} = \frac{-54\sqrt{3}+54}{9-3}$$

$$= \frac{-54\sqrt{3}+54}{6}$$

$$= -9\sqrt{3}+9$$

$$= 9(1-\sqrt{3})$$

$k = 9$

10

for upper bound of part c)

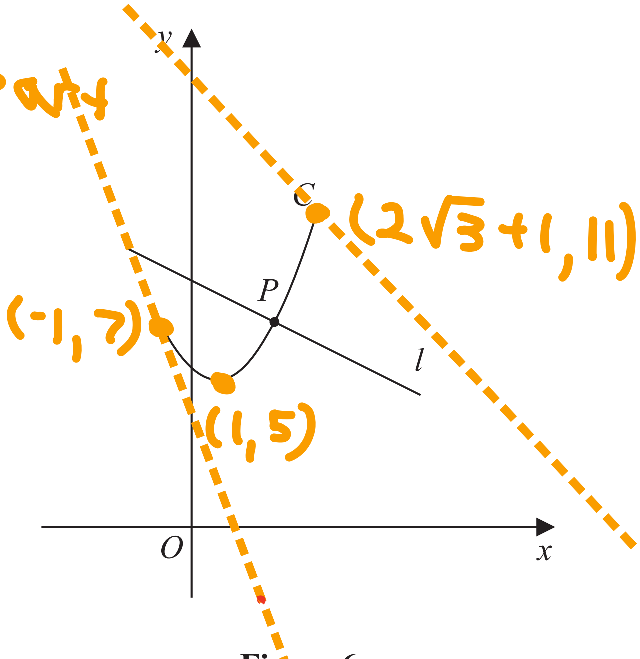


Figure 6

Figure 6 shows a sketch of the curve C with parametric equations

$$x = 2\tan t + 1 \qquad y = 2\sec^2 t + 3 \qquad -\frac{\pi}{4} \leqslant t \leqslant \frac{\pi}{3}$$

The line l is the normal to C at the point P where $t = \frac{\pi}{4}$

(a) Using parametric differentiation, show that an equation for l is

$$y = -\frac{1}{2}x + \frac{17}{2}$$

(5)

(b) Show that all points on C satisfy the equation

$$y = \frac{1}{2}(x-1)^2 + 5$$

(2)

The straight line with equation

$$y = -\frac{1}{2}x + k \qquad \text{where } k \text{ is a constant}$$

intersects C at two distinct points.

(c) Find the range of possible values for k.

(5)

P 6 9 6 0 2 A 0 4 4 4 8

a) $x = 2\tan t + 1$, $y = 2\sec^2 t + 3 = 2(\sec t)^2 + 3$

$\frac{dx}{dt} = 2\sec^2 t$, $\frac{dy}{dt} = 4(\sec t)\sec t \tan t$

$= 4\sec^2 t \tan t$

$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = 4\sec^2 t \tan t \times \frac{1}{2\sec^2 t}$

$= \frac{4\sec^2 t \tan t}{2\sec^2 t}$

$= 2\tan t$

When $t = \frac{\pi}{4}$: $\frac{dy}{dx} = 2\tan\frac{\pi}{4} = 2(1) = 2$

So $m = 2$

Normal has perpendicular gradient

$\Rightarrow m_\perp = -\frac{1}{2}$

Normal equation becomes

$y - y_1 = -\frac{1}{2}(x - x_1)$ ①

We need the point (x_1, y_1) still

When $t = \frac{\pi}{4}$: $x = 2\tan\frac{\pi}{4} + 1 = 3$

$y = 2\sec^2\frac{\pi}{4} + 3 = \frac{2}{\cos^2\frac{\pi}{4}} + 3$

$= \frac{2}{\left(\frac{\sqrt{3}}{2}\right)^2} + 3 = 7$

Question 16 continued

plug (3,7) into 1

$$y - 7 = -\frac{1}{2}(x-3)$$

$$y = -\frac{1}{2}x + \frac{3}{2} + 7$$

$$y = -\frac{1}{2}x + \frac{17}{2}$$

b) $x = 2\tan t + 1$, $y = 2\sec^2 t + 3 = 2(\sec t)^2 + 3$

Need to find the cartesian equation

$\tan t = \frac{x-1}{2}$ $\sec^2 t = \frac{y-3}{2}$

plug into quadratic identity $1 + \tan^2 t = \sec^2 t$

$$1 + \left(\frac{x-1}{2}\right)^2 = \frac{y-3}{2}$$

$$y - 3 = 2\left[1 + \left(\frac{x-1}{2}\right)^2\right]$$

$$y - 3 = 2 + \frac{(x-1)^2}{2}$$

$$y = \frac{(x-1)^2}{2} + 5 \Rightarrow y = \frac{1}{2}(x-1)^2 + 5$$

Question 16 continued

c) $y = \frac{(x-1)^2}{2} + 5$, $y = -\frac{1}{2}x + k$

$-\frac{1}{2}x + k = \frac{(x-1)^2}{2} + 5$

$-\frac{1}{2}x + k = \frac{x^2 - 2x + 1}{2} + 5$

$-x + 2k = x^2 - 2x + 1 + 10$

$x^2 - x + 11 - 2k = 0$

intersects at 2 distinct points

$\Rightarrow$ discriminant > 0 i.e. $b^2 - 4ac > 0$

$a = 1$

$b = -1$

$c = 11 - 2k$

$(-1)^2 - 4(1)(11-2k) > 0$

$1 - 44 + 8k > 0$

$8k > 43$

$k > \frac{43}{8}$

Also curve is parametric with $-\frac{\pi}{4} \leq t < \frac{\pi}{3}$

So curve doesn't continue forever

$x = 2\tan t + 1$, $y = 2(\sec t)^2 + 3$, $-\frac{\pi}{4} \leq t < \frac{\pi}{3}$

$2\tan\left(-\frac{\pi}{4}\right) + 1 = -1$ and $2\tan\left(\frac{\pi}{3}\right) + 1 = 2\sqrt{3} + 1$

$2\sec^2\left(-\frac{\pi}{4}\right) + 3 = 7$ and $2\sec^2\left(\frac{\pi}{3}\right) + 3 = 11$

$y = -\frac{1}{2}x + k$

plug in $(-1, 7)$ and $(2\sqrt{3} + 1, 11)$

$7 = -\frac{1}{2}(-1) + k$

$7 = \frac{1}{2} + k$

$k = \frac{13}{2}$

$11 = -\frac{1}{2}(2\sqrt{3} + 1) + k$

$22 = -2\sqrt{3} - 1 + 2k$

$2k = 22 + 2\sqrt{3} + 1$

$k = 13.2$

so need $k \leq \frac{13}{2}$ and $k \leq 13.2$

overlap is $k \leq \frac{13}{2}$ to make both true

We know $k > \frac{43}{8}$ already

$\frac{43}{8} < k \leq \frac{13}{2}$